UNIT 2

INTRODUCTION & BASIC CONCEPTS OF SET THEORY

SET THEORY

1. INTRODUCTION TO SET THEORY

A **set** is a well-defined collection of distinct objects.

Examples

- $A = \{1,2,3,4\}$
- $V = \{a,e,i,o,u\}$
- $N = \{1,2,3,...\}$

Notation

- Capital letters denote sets: A, B, C
- Small letters denote elements: a, b, c
- $\in$: belongs to
- $\notin$: does not belong to

Example:

- $3 \in \{1,2,3,4\}$
- $5 \notin \{1,2,3,4\}$

2-Mark Questions (Short Answer / Conceptual Frameworks)

Q. Define a Set.
A. A set is a well-defined collection of distinct objects. The objects belonging to a set are called its elements or members.

Q. What is meant by Cardinality of a set?
A. The total number of unique elements present in a set is called its cardinality. For a set A, it is denoted by $|A|$ or $n(A)$.

Q. Define an Empty Set.
A. A set containing no elements is called an empty set, null set, or void set. It is universally denoted by the symbol $\emptyset$ or $\{\}$.

Example: $A = \{x \mid x<0 \text{ and } x\in N\}$

Since no natural number is less than 0, $A = \emptyset$

Q. What is a Singleton Set?
A. A set containing exactly one single element is called a singleton set. Example: $A = \{5\}$.

Q. Define Power Set.
A. The set of all possible subsets of a given set A is called its Power Set, denoted by P(A). If $|A| = n$, then $|P(A)| = 2^n$.

Q. Define Equal Sets

A. Two sets are equal if they contain the same elements.

Example: $A=\{1,2,3\}$, $B=\{3,2,1\}$, $A=B$

Q. Equivalent Sets

A. Sets having the same number of elements.

Example: $A=\{1,2,3\}$, $B=\{a,b,c\}$

$n(A)=n(B)=3$.

Q. Universal Set

A. Contains all objects under consideration.

Notation: U

Example: $U=\{1,2,3,4,5,6\}$

Q. Differentiate between a Subset and a Proper Subset with an example.
A. A set B is a subset of A (denoted $B \subseteq A$) if every element of B is in A; this allows B to be identical to A. A set B is said to be a proper subset of A (denoted $B \subset A$) if every element of B is also an element of A, AND there is at least one element in A that is not present in B (i.e., $A \neq B$).
Example: If $A = \{1, 2, 3\}$ and $B = \{1, 2\}$, then B is a proper subset of A.

Q. Define the Relative Complement (Set Difference) of two sets with its algebraic rule.
A. Let U be the Universal Set. The relative complement of a set A with respect to B, denoted by B - A (or $B \setminus A$), is the set of all elements that belong to B but do not belong to A.
Formula: $B - A = \{ x \mid x \in B \text{ and } x \notin A \}$.

Q. State De Morgan's Laws for Set Operations.
A. According to De Morgan's Laws of Set Theory, for any two finite sets A and B:
1) $(A \cup B)' = A' \cap B'$ (The complement of the union is the intersection of the complements)
2) $(A \cap B)' = A' \cup B'$ (The complement of the intersection is the union of the complements).

Q. Explain Symmetric Difference of sets and give its formula.
A. The Symmetric Difference of two sets A and B, denoted by $A \Delta B$, is the set of elements that belong to either A or B, but not to both.
Algebraic Formulas: $A \Delta B = (A - B) \cup (B - A)$ or $A \Delta B = (A \cup B) - (A \cap B)$.

REPRESENTATION OF DISCRETE STRUCTURES

Q. What is meant by a Discrete Structure?
A. A discrete structure is an abstract mathematical model that deals with distinct, separated values or structures rather than continuous ranges (e.g., graphs, networks, finite sets).

Q. Define the Set-Builder Method of representation.
A. A Set-Builder representation defines a set by specifying a property or rule that all its members must uniquely satisfy. Syntax: $A = \{ x \mid P(x) \}$.

Q. What is a Venn Diagram?
Ans: A Venn Diagram is a visual, pictorial representation of sets using closed geometric curves (usually circles) inside a rectangle representing the universal set.

Q. State the two primary mathematical notations used to describe sets with an example of each.
A. 1) Roster / Tabular Form: Explicitly listing every element inside curly brackets separated by commas. Example: $V = \{a, e, i, o, u\}$.
2) Set-Builder Form: Using a logical predicate defining property. Example: $V = \{x \mid x \text{ is a vowel in the English alphabet}\}$.

Q. Explain the Bit-String method for representing sets within computer memory structures.
A. A Computer Bit-String representation maps a universal set U of n elements to an ordered sequence of n bits. If the i-th element of U belongs to a subset A, the i-th bit is set to 1; otherwise, it is set to 0. Example: If $U = \{1, 2, 3, 4, 5\}$ and $A = \{1, 3, 5\}$, the bit string representation of A is 10101.

RELATIONS AND ORDERING

Q. Define Cartesian Product of two sets.
A. The Cartesian Product of two sets A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. Formula: $A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$.

Q. Define a Binary Relation.
A. A binary relation R from set A to set B is a subset of the Cartesian product $A \times B$. That is, $R \subseteq A \times B$.

Q. What is a Reflexive Relation?
A. A relation R on a set A is reflexive if every element is related to itself. Formally, $\forall a \in A, (a, a) \in R$.

Q. Define Equivalence Relation.
A. A binary relation R on a set A is an equivalence relation if it is simultaneously Reflexive, Symmetric, and Transitive.

Q. What is a POSET?
A. A set paired with a Partial Ordering Relation (POSET) is called a Partially Ordered Set, denoted as $(A, \leq)$.

Q. If set A has m elements and set B has n elements, calculate the total number of relations possible from A to B.
A. If set A has 'm' elements and set B has 'n' elements, the cardinality of their Cartesian product is $|A \times B| = m \times n$. Since a relation is any subset of $A \times B$, the total number of unique relations possible is $2^{(m \times n)}$.

Q. Clearly distinguish between Symmetric and Anti-Symmetric relations.
A. A relation R on a set A is symmetric if whenever $(a, b) \in R$, then $(b, a) \in R$. It is anti-symmetric if whenever $(a, b) \in R$ and $(b, a) \in R$, then $a = b$.
Example: '=' is symmetric and anti-symmetric; '<=' is anti-symmetric but not symmetric.

Q. What is the difference between a Partial Ordering relation and an Equivalence relation?
A. A Partial Ordering relation is Reflexive, Anti-Symmetric, and Transitive. An Equivalence relation is Reflexive, Symmetric, and Transitive. The key difference lies in Anti-Symmetry versus Symmetry.

FUNCTIONS

Q. Define a Function in discrete mathematics.
A. A function f from set A to set B (denoted $f: A \to B$) is a special relation that assigns to each element $x \in A$ exactly one unique element $y \in B$.

Q. What are the Domain and Codomain of a function?
A. If $f: A \to B$, the set A is called the Domain, and the set B is called the Codomain of the function f.

Q. Define the Range of a function.
A. The Range of a function is the set of all actual images of elements of the domain.

Range = $\{ f(x) \mid x \in A \} \subseteq$ Codomain.

Q. Define Injection and Surjection mapping criteria.
A. A function $f: A \to B$ is an Injection (One-to-One) if distinct elements in the domain map to distinct elements in the codomain. Formally: $f(x) = f(y) \Rightarrow x = y$.
It is a Surjection (Onto) if every element in the codomain has at least one pre-image in the domain. Formally: Range = Codomain.

Q. What is a Bijection? Why is it crucial for inverse functions?
A. A function is called a Bijection if it is simultaneously One-to-One (Injection) and Onto (Surjection). Only bijective functions possess a well-defined Inverse Function (f^-1).

Q. Define Composition of Functions.
A. Let $f: A \to B$ and $g: B \to C$. The composite function, denoted by $(g \circ f): A \to C$, is defined by the algebraic execution sequence $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

5-Mark Questions

Q. Problem Statement: In an examination of a class of 100 engineering students, 72 students passed in Data Structures (DS) and 43 students passed in Discrete Mathematics (DM). Find:
a) The number of students who passed both subjects.
b) The number of students who passed DS only.
c) The number of students who passed DM only.

Solution. Let S_DS be the set of students who passed Data Structures, and S_DM be the set who passed Discrete Mathematics.
Total unique students $n(S_{DS} \cup S_{DM}) = 100$
Given values: $n(S_{DS}) = 72$, $n(S_{DM}) = 43$

a) Finding students who passed both (Intersection): Formula: $n(S_{DS} \cup S_{DM}) = n(S_{DS}) + n(S_{DM}) - n(S_{DS} \cap S_{DM})$
$100 = 72 + 43 - n(S_{DS} \cap S_{DM})$
$100 = 115 - n(S_{DS} \cap S_{DM})$
$n(S_{DS} \cap S_{DM}) = 115 - 100 = 15$ students.
Therefore, 15 students passed both subjects.

b) Finding students who passed DS only: $n(S_{DS}\ \text{Only}) = n(S_{DS}) - n(S_{DS} \cap S_{DM}) = 72 - 15 = 57$ students.

c) Finding students who passed DM only: n(S_DM Only) = n(S_DM) - n(S_DS $\cap$ S_DM) = 43 - 15 = 28 students.
Verification: 57 (DS only) + 28 (DM only) + 15 (Both) = 100 total students. The answer is verified.

> **JNTU Exam Tip:** The principle used above is called the Principle of Inclusion-Exclusion (PIE). For three sets, the extended formula is: $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$.

Q. Prove the Distributive Law of Intersection over Union: Prove analytically using set laws that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof: To prove that the LHS set equals the RHS set, we must show that LHS $\subseteq$ RHS and RHS $\subseteq$ LHS.
Step 1: Prove LHS $\subseteq$ RHS
Let x be an arbitrary element such that $x \in A \cap (B \cup C)$.
$\rightarrow$ $x \in A$ AND $x \in (B \cup C)$ (By definition of Intersection)
$\rightarrow$ $x \in A$ AND ($x \in B$ OR $x \in C$) (By definition of Union)
$\rightarrow$ ($x \in A$ AND $x \in B$) OR ($x \in A$ AND $x \in C$) (By logical distribution of 'AND' over 'OR')
$\rightarrow$ ($x \in A \cap B$) OR ($x \in A \cap C$) (By definition of Intersection)
$\rightarrow$ $x \in (A \cap B) \cup (A \cap C)$ (By definition of Union)
Since any x in LHS is also in RHS, LHS $\subseteq$ RHS.

Step 2: Prove RHS $\subseteq$ LHS
Let y be an arbitrary element such that $y \in (A \cap B) \cup (A \cap C)$.
$\rightarrow$ $y \in (A \cap B)$ OR $y \in (A \cap C)$ (By definition of Union)
$\rightarrow$ ($y \in A$ AND $y \in B$) OR ($y \in A$ AND $y \in C$) (By definition of Intersection)
$\rightarrow$ $y \in A$ AND ($y \in B$ OR $y \in C$) (By factoring out logical 'AND $y \in A$')
$\rightarrow$ $y \in A$ AND $y \in (B \cup C)$ (By definition of Union)
$\rightarrow$ $y \in A \cap (B \cup C)$ (By definition of Intersection)
Since any y in RHS is also in LHS, RHS $\subseteq$ LHS.

Conclusion: Because LHS $\subseteq$ RHS and RHS $\subseteq$ LHS, it is conclusively proven that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Q.E.D.

5-Mark Questions

Q1. Comprehensive Problem on Bit-String Arithmetic: Let the Universal Set be U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. Subset A = {1, 3, 5, 7, 9} and Subset B = {2, 3, 5, 7}.
a) Represent sets A and B as Bit Strings based on the order of elements in U.
b) Perform Bitwise operations (AND, OR, NOT) to find the bit strings for $A \cap B$, $A \cup B$, and A'.
c) Decode the resulting bit strings back into standard roster set notation.

Solution:
a) Bit String Generation: The universe has n=10 elements. The position corresponds to numbers 1 through 10.
Subset A contains elements 1, 3, 5, 7, 9 $\rightarrow$ Bit String A = 1 0 1 0 1 0 1 0 1 0
Subset B contains elements 2, 3, 5, 7 $\rightarrow$ Bit String B = 0 1 1 0 1 0 1 0 0 0

b) Calculation of Intersection (A ∩ B) via Bitwise AND: Bit String A: 1 0 1 0 1 0 1 0 1 0
Bit String B: 0 1 1 0 1 0 1 0 0 0

Bitwise AND: 0 0 1 0 1 0 1 0 0 0 (Positions 3, 5, 7 are 1)
Decoding A ∩ B: Elements corresponding to 1s are {3, 5, 7}.

c) Calculation of Union (A ∪ B) via Bitwise OR: Bit String A: 1 0 1 0 1 0 1 0 1 0
Bit String B: 0 1 1 0 1 0 1 0 0 0

Bitwise OR: 1 1 1 0 1 0 1 0 1 0 (Positions 1, 2, 3, 5, 7, 9 are 1)
Decoding A ∪ B: Elements corresponding to 1s are {1, 2, 3, 5, 7, 9}.

d) Calculation of Complement (A') via Bitwise NOT: Bit String A: 1 0 1 0 1 0 1 0 1 0

Bitwise NOT: 0 1 0 1 0 1 0 1 0 1 (Positions 2, 4, 6, 8, 10 are 1)
Decoding A': Elements corresponding to 1s are {2, 4, 6, 8, 10}.

5-Mark Questions

Q. Problem Statement: Let Z be the set of all integers. A relation R is defined on Z such that R = { (x, y) | x - y is a multiple of 5 }. Prove rigorously that R is an Equivalence Relation.

Proof:
To prove that R is an equivalence relation, we must verify three fundamental structural properties:

1. Reflexivity: We must check if (x, x) ∈ R for all integers x.
Formally, evaluate the difference: x - x = 0.
Since 0 can be written as 5 × 0, it is an integer multiple of 5. Hence, (x, x) ∈ R. R is Reflexive.

2. Symmetry: Assume that (x, y) ∈ R. This implies that x - y is a multiple of 5.
Therefore, we can write: x - y = 5k, where k is an arbitrary integer.
Factoring out a negative sign: y - x = -(x - y) = -(5k) = 5(-k).
Since k is an integer, -k is also an integer. Thus, y - x is an integer multiple of 5. This implies (y, x) ∈ R.
R is Symmetric.

3. Transitivity: Assume that (x, y) ∈ R and (y, z) ∈ R. This means:
x - y = 5m (for some integer m)
y - z = 5n (for some integer n)
Add the two linear algebraic equations together:
(x - y) + (y - z) = 5m + 5n
x - z = 5(m + n)
Since m and n are integers, their sum (m + n) is also an integer. Therefore, x - z is a multiple of 5, which means (x, z) ∈ R. R is Transitive.

Conclusion: Since the relation R is Reflexive, Symmetric, and Transitive simultaneously, it is proven to be an Equivalence Relation over the integers. Q.E.D.

Q. Problem Statement on Partially Ordered Sets: Let A = {1, 2, 3, 4, 6, 12} and let R be the partial ordering relation 'divides' (denoted by |) such that xRy means x divides y exactly without a remainder.
a) Construct the structural Relation Matrix M_R.
b) Draw the Hasse Diagram for this POSET.

Solution: Let's map out the ordered pairs belonging to R:
R = {(1,1),(1,2),(1,3),(1,4),(1,6),(1,12), (2,2),(2,4),(2,6),(2,12), (3,3),(3,6),(3,12), (4,4),(4,12), (6,6),(6,12), (12,12)}

|	1	2	3	4	6	12
1	1	1	1	1	1	1
2	0	1	0	1	1	1
3	0	0	1	0	1	1
4	0	0	0	1	0	1
6	0	0	0	0	1	1
12	0	0	0	0	0	1

b) Hasse Diagram Construction Description: To draw the Hasse Diagram, remove all reflexive self-loops and eliminate transitive implied edges (e.g., since 1|2 and 2|4, eliminate line 1-4).
Levels of the diagram from bottom to top based on structural dependencies:
Level 1 (Bottom baseline element): 1 (divides everything)
Level 2: Elements 2 and 3 (drawn above 1, with lines connecting 1→2 and 1→3)
Level 3: Elements 4 and 6. Draw 4 directly above 2 (connect 2→4). Draw 6 above 2 and 3 (connect 2→6 and 3→6).
Level 4 (Top elements): Element 12. Draw 12 at the peak, connecting 4→12 and 6→12.

5-Mark Questions

Q1. Analytical Problem Statement: Let f: R → R be defined by f(x) = 2x + 3 and g: R → R be defined by g(x) = x^2 - 2.
a) Determine the operational explicit formulas for the composite functions (g ∘ f)(x) and (f ∘ g)(x).
b) Prove whether function f(x) is a bijective function, and if so, derive its inverse function formula f^-1(x).

Solution:
a) Computing Composition (g ∘ f)(x): (g ∘ f)(x) = g(f(x)) = g(2x + 3)
Substitute (2x + 3) into the variable slot of g(x):
(g ∘ f)(x) = (2x + 3)^2 - 2
Expand the algebraic binomial square: = 4x^2 + 12x + 9 - 2
(g ∘ f)(x) = 4x^2 + 12x + 7

b) Computing Composition (f ∘ g)(x): (f ∘ g)(x) = f(g(x)) = f(x^2 - 2)
Substitute (x^2 - 2) into the variable slot of f(x):
(f ∘ g)(x) = 2(x^2 - 2) + 3
Distribute the multiplication: = 2x^2 - 4 + 3
(f ∘ g)(x) = 2x^2 - 1
Note: (g ∘ f)(x) ≠ (f ∘ g)(x), proving composition is non-commutative.

c) Bijective Proof and Inverse Derivation for $f(x) = 2x + 3$: 1. Check Injection (One-to-One): Let
$f(x_1) = f(x_2)$
$2(x_1) + 3 = 2(x_2) + 3$
Subtract 3 from both sides: $2(x_1) = 2(x_2) \Rightarrow x_1 = x_2$. Thus, f is Injectivity compliant.

2. Check Surjection (Onto): Let y be an arbitrary element in codomain R. Set $y = f(x)$:
$y = 2x + 3 \Rightarrow y - 3 = 2x \Rightarrow x = (y - 3) / 2$
For every real number y, x is a real number. Thus, a valid pre-image exists for every y. f is Surjective.
Conclusion: Since f is both injective and surjective, it is Bijective.

3. Deriving Inverse: From the surjection step, $x = f^{-1}(y) = (y - 3) / 2$.
Replacing variable format into standard convention: $f^{-1}(x) = (x - 3) / 2$.